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LAMINAR-TURBULENT TRANSITION: A THEORETICAL MODEL OF THE NONLINEAR EVOLUTION OF THREE DIMENSIONAL WAVETRAINS IN BOUNDARY LAYERS

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Summary. *A weakly nonlinear model of the evolution of three-dimensional wavetrains in boundary layers is presented. This was motivated by wind tunnel experiments with wavetrains emanating from a point source. The experiments that showed that the first signature of nonlinearity was the appearance of secondary mean flow distortions. The distortions were in the form of longitudinal streaks with a spanwise structure which increased in complexity with the downstream evolution. The model linked the streaks with a pair counter-rotating longitudinal vorticies that distort the base mean flow by a litf-up mechanism of redistribution of streamwise momentum. The more complicated structures developed further downstream are also addressed. They are explained by a second order correction to the weakly nonlinear model. The results show qualitative agreement with the experimental observations.*

Key-words: *Hydrodynamic instability, laminar-turbulent transition, boundary layers, weakly nonlinear theory, perturbation methods*

1. INTRODUCTION

The laminar-turbulent transition process that occurs naturally often involves oscillations that are highly three-dimensional. Despite its practical importance, the mechanisms involved are not fully understood, and further investigation is needed into the nonlinear evolution of such disturbances. The current paper presents a theoretical model for the nonlinear evolution of a wavetrain of small amplitude emanating from a point source in a flat plate boundary layer.

The study has its roots in the work of Gaster who carried out an experimental investigation of the linear and nonlinear evolution of a wavepacket in a flat plate boundary layers (Gaster and Grant, 1975) and developeded theoretical model for the linear stage (Gaster, 1975). More recently Medeiros (1996) and Medeiros and Gaster (1999a, 1999b) carried out further experimental, numerical and theoretical work in the subject. The investigation concluded that the early stages of the nonlinear evolution of wavepackets could not be entirely explained by the secondary instability mechanisms that have been observed in the nonlinear regime of plane regular wavetrains. The investigation has also shown via numerical simulations that the nonlinear mechanism involved in the evolution of the wavepackets are essentially three-dimensional. The effort, however, could not provide a theoretical model of the nonlinear evolution of wavepackets. It appeared that a simpler three-dimensional wave-systems, namely, a three-dimensional wavetrain emanating from a point source, could shed some light into the more complicated three-dimensional wavepacket.

The feature of the nonlinear evolution of wavetrains with which this paper is concerned is the appearance of longitudinal streaks which have been observed experimentally . Initially the spanwise structure of the streaks is relatively simple, but further downstream it becomes more complicated, involving the splitting of streaks. The weakly nonlinear model here presented links the phenomenon to a nonlinear generation of longitudinal vortices which would then produce a mean flow distortion throught a lift-up mechanism of redistribution of streamwise momentum. The development of more complicated spanwise structures are also addressed here, and are linked to a second order approximation to the weakly nonlinear model.

2. BRIEF REVIEW OF EXPERIMENTAL RESULTS

Experimental results of the nonlinear evolution of three-dimensional wavetrains have been presented elsewhere. (Medeiros, 1998). To make this paper more readable some of the more important conclusions are presented here.

The experiments were carried out in the low turbulence wind tunnel of the University of Cambridge [1]. The laminar boundary layer with a free-stream velocity of 17m/s developed over a 1.68m long flat plate. The disturbances were produced by a loudspeaker embedded in the plate. They were introduced into the flow via a .3mm hole in the plate. The streamwise velocity records were obtained with a constant temperature hot-wire anemometer mounted on a computer controlled traverse gear.

The disturbance introduced was not a continuous wavetrain. A finite wave-

[1]Now located at Queen Mary & Westfield College - London

train was used instead, but tests were carried out to ensure that it was long enough so that its central part behaved like a continuous one. The passage of the finite wavetrain can be treated as an event. The ensemble average of 128 events was taken at each measuring station resulting in a clearer signal. The first non-linear signature of the streamwise evoltuion of the wavetrain was the appearance of a mean flow distortion, see figure 2 of Medeiros (1998). One interesting feature of the evolution of this distortion was that initially it was negative, but changed to positive at latter stages. The distortion appeared to have little effect on the evolution of the wavetrain which grew and decayed in a way that was consistent with the linear instability theory.

Measurements were also carried out at a number of points across the span, building a three-dimensional picture of the wave system, see figure 3 of Medeiros (1998). The evolution of the mean flow distortion field was shown by contour plots in $t-z$ planes at a number of downstream positions. It was observed that the mean flow distortion formed longitudinal streaks. It can also be seen that the change of sign from negative to positive mean distortion observed was related to the growth of a positive streak at the center of a negative streak.

3. EARLY NONLINEAR STAGE

The equations of motion for incompressible flow read:

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \frac{1}{R}\nabla^2 \mathbf{v} \tag{2}$$

For a weakly nonlinear analysis we follow (Benney, 1961) and expand the velocity and the pressure fields as perturbation series:

$$\mathbf{v} = \mathbf{v}_0 + \epsilon \mathbf{v}_1 + \epsilon^2 \mathbf{v}_2, \tag{3}$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2, \tag{4}$$

where $\mathbf{v}_0 = (u_0(y), 0, 0)$ and p_0 represent the base flow; and ϵ is a measure of the amplitude of the primary Tollmien-Schlichting oscillation.

Although the experiments display spatial evolution, the analysis is simplified if one considers temporal instability. The model can be extended to spatial instability, but it has not been carried out at this stage. Therefore, hereafter the wavenumber α will, be taken as real. In a temporal instability model the disturbances are considered periodic in x. For the three-dimensional wavetrain all the modes have identical α which is linked to the frequency of the point source in the spatial case. The primary oscillation can then be represented by:

$$\mathbf{v}_1 = \mathbf{v}_{11}(y,z,t)e^{i\alpha x} + \mathbf{v}_{11}^*(y,z,t)e^{-i\alpha x}, \tag{5}$$

$$p_1 = p_{11}(y,z,t)e^{i\alpha x} + p_{11}^*(y,z,t)e^{-i\alpha x}. \tag{6}$$

In the expression, $*$ stands for complex conjugate.

Substituting the perturbation series with x-periodicity into the equations of motion (1) and (2)and collecting terms of order ϵ one arrives at:

$$\frac{\partial u_{11}}{\partial t} + i\alpha u_{00}u_{11} + v_{11}\frac{du_{00}}{dy} = -i\alpha p_{11} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2\right)u_{11}, \tag{7}$$

$$\frac{\partial v_{11}}{\partial t} + i\alpha u_{00}v_{11} = -\frac{\partial p_{11}}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2\right)v_{11}, \tag{8}$$

$$\frac{\partial w_{11}}{\partial t} + i\alpha u_{00}w_{11} = -\frac{\partial p_{11}}{\partial z} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2\right)w_{11}, \tag{9}$$

$$i\alpha u_{11} + \frac{\partial v_{11}}{\partial y} + \frac{\partial w_{11}}{\partial z} = 0. \tag{10}$$

Within a weakly nonlinear approach the secondary motion will be composed of a mean flow distortion and a second harmonic oscillation (Craik, 1985):

$$\mathbf{v}_2 = \mathbf{v}_{20}(y,z,t) + \mathbf{v}_{22}(y,z,t)e^{2i\alpha x} + \mathbf{v}_{22}^*(y,z,t)e^{-2i\alpha x}, \tag{11}$$

$$p_2 = p_{20}(y,z,t) + p_{22}(y,z,t)e^{2i\alpha x} + p_{22}^*(y,z,t)e^{-2i\alpha x}. \tag{12}$$

Substituting into the equations of motion and collecting terms both of order ϵ^2 ad independent of X yields:

$$\frac{\partial u_{20}}{\partial t} + v_{20}\frac{du_{00}}{dy} + \frac{\partial}{\partial y}\left(u_{11}v_{11}^* + u_{11}^*v_{11}\right) + \frac{\partial}{\partial z}\left(u_{11}w_{11}^* + u_{11}^*w_{11}\right) = \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_{20}, \tag{13}$$

$$\frac{\partial v_{20}}{\partial t} + 2\frac{\partial}{\partial y}\left(v_{11}^*v_{11}\right) + \frac{\partial}{\partial z}\left(v_{11}w_{11}^* + v_{11}^*w_{11}\right) = -\frac{\partial p_{20}}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v_{20}, \tag{14}$$

$$\frac{\partial w_{20}}{\partial t} + \frac{\partial}{\partial y}\left(v_{11}w_{11}^* + v_{11}^* w_{11}\right) + 2\frac{\partial}{\partial z}\left(w_{11}^* w_{11}\right) = -\frac{\partial p_{20}}{\partial z} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) w_{20}, \tag{15}$$

$$\frac{\partial v_{20}}{\partial y} + \frac{\partial w_{20}}{\partial z} = 0. \tag{16}$$

In the expressions, the first subscript corresponds to the order of approximation whereas the second represents the order of the harmonic.

Within a temporal instability analysis the primary oscillation takes the form

$$\mathbf{v}_{11} = \hat{\mathbf{v}}_{11}(y,z)e^{-i\alpha ct} \tag{17}$$

where c is the phase velocity.

In the equation governing the secondary mean motion in the streamwise direction (13), the third and fourth terms arise from the Reynolds stresses which were neglected in the equations governing the primary oscillations. The second term corresponds to the lift-up mechanism by which the streamwise momentum of the base flow is redistributed by convection with velocity v_{20}. For high Reynolds numbers we drop the viscous term and arrive at a solution:

$$u_{20} = -\hat{v}_{20}\frac{du_{00}}{dy}\frac{e^{2\alpha c_i t} - 1 - 2\alpha c_i t}{(2\alpha c_i)^2} + \hat{R}_{20}\frac{e^{2\alpha c_i t} - 1}{2\alpha c_i}, \tag{18}$$

$$\hat{R}_{20}(y,z) = -\frac{\partial}{\partial y}(\hat{u}_{11}\hat{v}_{11}^* + \hat{u}_{11}^*\hat{v}_{11}) + -\frac{\partial}{\partial z}(\hat{u}_{11}\hat{w}_{11}^* + \hat{u}_{11}^*\hat{w}_{11}), \tag{19}$$

$$v_{20}(y,z,t) = \hat{v}_{20}(y,z)e^{-2i\alpha ct}. \tag{20}$$

The double exponential growth shows that the secondary mean motion will eventually transcend to first order. For small values of c_i we find that the first term of the expression grows as t^2 whereas the second grows as t, indicating that for large times the lift-up mechanism is dominant over the forcing by the Reynolds stresses. The spanwise structure of the resulting u_{20}-field is, therefore, given by $\hat{v}_{20}$. To find $\hat{v}_{20}$ one has to solve the system of coupled equations (13) – (16).... for given primary oscillations. However, the $\hat{v}_{20}$-field is associated with a longitudinal vorticity field. It is easier to solve the equations for the vorticity field, for which one can find an uncoupled equation.

The second order mean longitudinal vorticity equation is obtained from equations (14) and (15)

$$\frac{\partial \xi_{20}}{\partial t} + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\left(v_{11}w_{11}^* + v_{11}^* w_{11}\right) + 2\frac{\partial^2}{\partial y \partial z}\left(w_{11}w_{11}^* - v_{11}v_{11}^*\right) = 0, \tag{21}$$

where we have neglected the viscous term. The solution also grows in time as a double exponential

$$\frac{\partial \xi_{20}}{\partial t} = \hat{S}_{20}(y,z)e^{2\alpha c_i t}, \tag{22}$$

$$\hat{S}_{20}(y,z) = \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)(\hat{v}_{11}\hat{w}_{11}^* + \hat{v}_{11}^*\hat{w}_{11}) + 2\frac{\partial}{\partial y \partial z}(\hat{w}_{11}\hat{w}_{11}^* - \hat{v}_{11}\hat{v}_{11}^*) . \tag{23}$$

The spanwise structure is, therefore, given by $\hat{S}_{20}$ which can be calculated from the primary Tollmien-Schlichting oscillation governed by equations (7)–(10).

Equations (7)–(10) are in separable form and permit the normal modes solution

$$\mathbf{v}_{11}(x,y,z,t) = \tilde{\mathbf{v}}_{11}(y)e^{i(\alpha x + \beta z - \alpha c t)} \tag{24}$$

$$p_{11}(x,y,z,t) = \tilde{p}_{11}(y)e^{i(\alpha x + \beta z - \alpha c t)}, \tag{25}$$

where β is the spanwise wavenumber of the mode.

Substituting into equations (7)–(10) yields (Mack ,1984):

$$i(\alpha \tilde{u}_{00} - \alpha c)\tilde{u}_{11} + \tilde{v}_{11}\frac{\partial u_{00}}{\partial y} = i\alpha \tilde{p}_{11} + \frac{1}{R}\left[\frac{\partial^2}{\partial y^2} - (\alpha^2 + \beta^2)\right]\tilde{u}_{11} \tag{26}$$

$$i(\alpha \tilde{u}_{00} - \alpha c)\tilde{u}_{11} = -\frac{\partial \tilde{p}_{11}}{\partial y} + \frac{1}{R}\left[\frac{\partial^2}{\partial y^2} - (\alpha^2 + \beta^2)\right]\tilde{v}_{11} \tag{27}$$

$$i(\alpha \tilde{u}_{00} - \alpha c)\tilde{v}_{11} = i\beta \tilde{p}_{11} + \frac{1}{R}\left[\frac{\partial^2}{\partial y^2} - (\alpha^2 + \beta^2)\right]\tilde{w}_{11} \tag{28}$$

$$i(\alpha \tilde{u}_{11} + \beta \tilde{w}_{11}) + \frac{\partial \tilde{v}_{11}}{\partial y} = 0, \tag{29}$$

with Dirichlet boundary conditions for u, v and w at both $y = 0$ and $y = \infty$: This system of equations is very stiff and presents some difficulties to be solved. Several techniques can be used to overcome these difficulties. In the current implementation a filter technique developed by Kaplan (Mack, 1984) was used.

The equations also permit the simulation of the linear evolution of the disturbances. Assuming that all modes are equally excited at the origem it is possible to determine the flow field at a given position in time or space depending on whether temporal or spatial instability is considered. However, it is known from experimental (Medeiros, 1998) and theoretical evidence that, because the boundary layer acts as a filter, the initial flat spetrum will, at later stages of the evolution, result in a spectral distribution that resembles a gaussian curve. We have therefore modelled the three-dimensional wavetrain as a number of modes with identical streamwise wavenumber and different spanwise wavenumbers; and with a gaussian distribution of spectral energy centred on the mode with zero spanwise wavenumber.

The eigenvectors of these modes were calculated from equations (26) and (29) and an inverse Fourier transform in the spanwise direction was applied. The resulting $y - z$ structure of the three-components of the velocity field are shown in figures 1 and 2. These velocities allow the calculation of $\hat{S}(y, z)$ which gives the structure of the longitudinal vorticity , figure 3

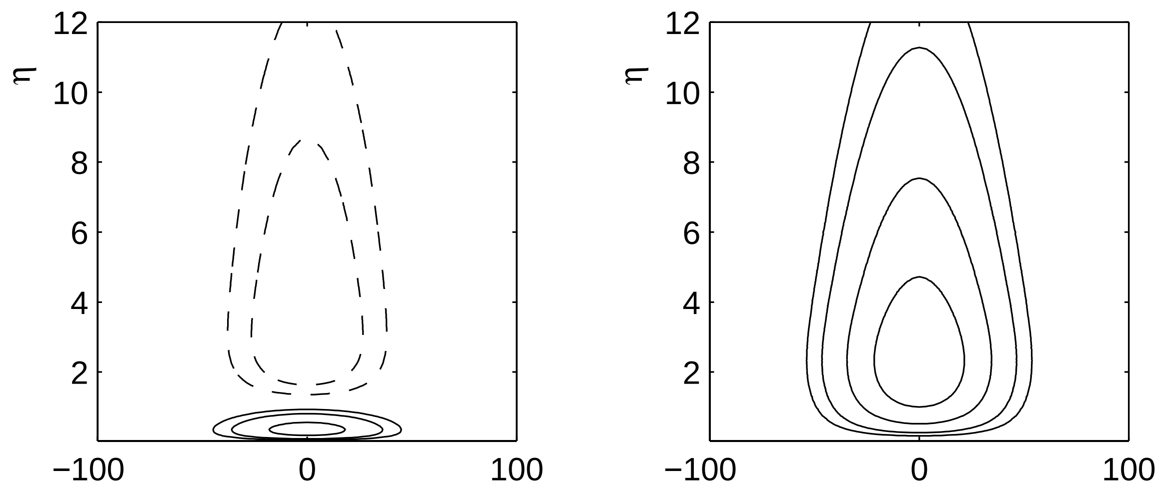

Figura 1: Vertical velocity distribution on a $y - z$ plane. Left frame, real part. Contour levels: $(1,.5,.2,.1)\times10^{-3}$. Right frame, imaginary part. Contour levels: $(1,.5,.2,.1)\times10^{-2}$.

Figure 3 shows the existence of a pair of counter-rotating vortices. It is clear that the vertical velocity field associated with this vorticity distribution consists of a region of positive velocity in the region between the vortices and positive outside the vortices, with a slow decay for larger values of $|z|$. The same structure will be imposed on the mean streamwise velocity field through the redistribution of momentum by the lift-up mechanism, equation (13). The positive vertical velocity will produce a negative mean flow distortion whereas the negative vertical velocity will produce positive distortions. The streak formation at x=800mm in figure 3 of Medeiros (1998) is consistent with the existence of two counter-rotating vortices.

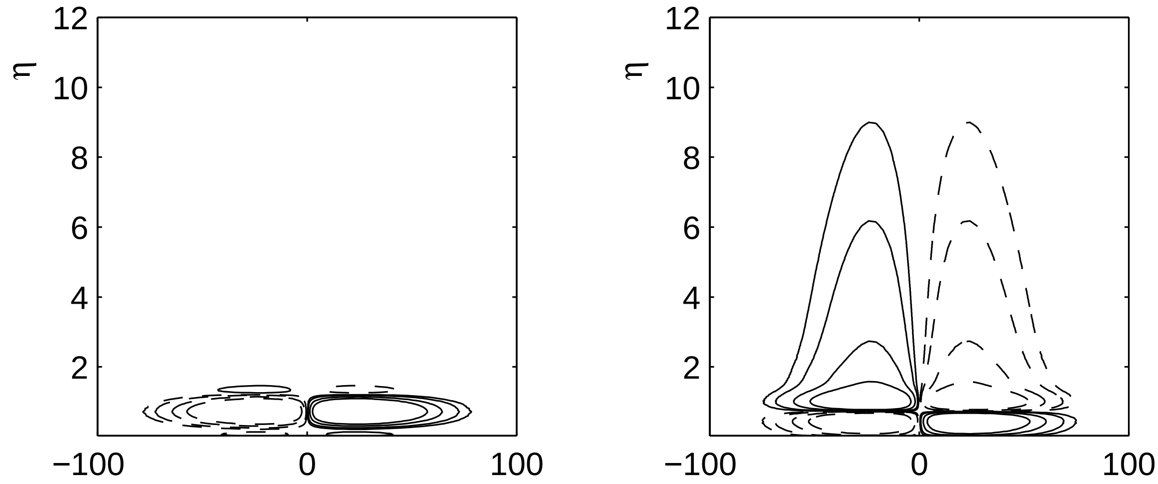

Figura 2: Vertical velocity distribution on a $y - z$ plane. Left frame, real part. Right frame, imaginary part. Contour levels: $(1,.5,.2,.1)\times 10^{-3}$

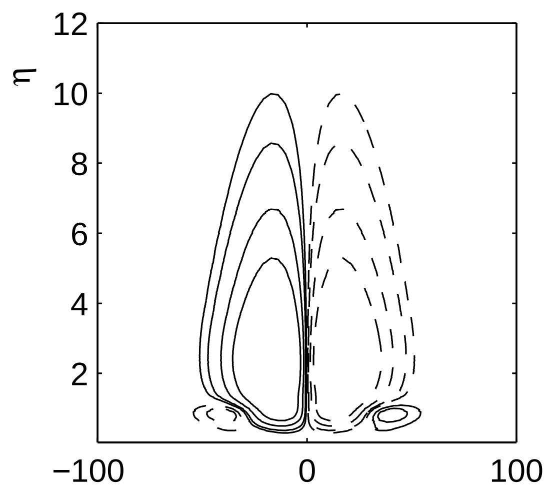

Figura 3: Longitudinal vorticity distribution on a $y - z$ plane. Contour levels: $(1,.5,.2,.1)\times 10^{-5}$

4. LATE NONLINEAR STAGE

The lift-up mechanism described by equation (13) does not consider that as the secondary mean motion builds up the base flow is distorted and at some stage the streamwise momentum that is redistributed is no longer that of the original base flow, but of a distorted spanwise-dependent profile. Since the secondary mean motion grows as a double exponential, this effect should eventually become important. If one wants to include such effects equation (30) should read:

$$\frac{\partial u_{20}}{\partial t} + v_{20}\frac{du_{00}}{dy} + \epsilon v_{20}\frac{\partial u_{20}}{\partial y} + \epsilon w_{20}\frac{\partial u_{20}}{\partial z} + \hat{R}_{20}(y,z)e^{2\alpha c_i t} = 0, \tag{30}$$

with viscous terms neglected. Just as equation (13), this equation is linear and describes the convection of the streamwise momentum as a passive scalar.

Analysis of equation (30) for a three-dimensional wavetrain is quite involving because a large number of spanwise modes has to be considered. However, the basic features can be illustrated by considering a single three-dimensional mode. In that case the velocity field induced by the streamwise vorticity would be

$$v_{20}(y,z) = \tilde{v}_{20}(y)\cos\beta z \qquad w_{20}(y,z) = \frac{-1}{\beta}\frac{d\tilde{v}_{20}(y)}{dy}\sin\beta z. \tag{31}$$

The streamwise mean motion would then develop a spanwise structure

$$\begin{aligned}\hat{u}_{20}(y,z) = \tilde{u}_{200}(y) + \tilde{u}_{201}(y)\cos(\beta z) + \tilde{u}_{202}(y)\cos(2\beta z)\\ +\tilde{u}_{203}(y)\cos(3\beta z) + ...,\end{aligned} \tag{32}$$

where the third subscript identifies the order of the term in the series. A study of the convergence of the series for Couette base flow showed that about 11 terms are needed (wallefe, 1995). Such analysis has not yet been carried for a Blasius profile, but the model clearly leads to the appearance of the higher spanwise modes of mean flow distortion. This is consistent with the appearance of other streaks at later stages of the nonlinear evolution.

5. DISCUSSION

Ths paper presents a weakly nonlinear model for the evolution of three-dimensional wavetrains of small amplitude in boundary layers. The model focuses on the generation of longitudinal streaks that have been observed in experiments. The nonlinear regime is divided into two stages. The early stage corresponds to the appearance of the first streaks. The later stage involves the appearance of other streaks that split the previous ones into a larger number.

The origem of the streaks is explained by the forcing of longitudinal vorticity by the nonlinear interaction between the vertical and tranversal primary motion. The model predicts a vorticity distribution that is consistent, in both z and y directions, with the streak structure. In particular, the model predicts the appearance of a low velocity streak at the center of the disturbance field and high velocity streaks on each side of the disturbance field. In addition, the model predicts that the mean flow distortion is localized inside the boundary layer, which is in agreement with the experiments.

The late stage was linked to a second order correction to the nonlinear model. It predicts the generation of higher spanwise harmonics which is consistent with the appearance of other streaks.

It is known that longitudinal streaks are one of the ingredients of turbulent flow. It is conjectured that these results might provide a short cut between the early wave-like behaviour and the vortical structures observed in turbulent flow. This short cut would take place for highly three-dimensional waves.

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